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UNCOUNTABLY MANY CONTRACTIBLE OPEN 4-MANIFOLDS

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§1. INTRODUCTION

McMILLAN [6] has constructed uncountably many contractible open 3-manifolds and Curtis and Kwun [1] have shown for each $n \geq 5$ there exist uncountable many contractible open n-manifolds.

Here we will take care of the remaining case where $n = 4$ by generalizing Mazur's embedding of the dunce hat in S^4 (refer to [5, 9]) to get countably many different contractible 4-manifolds with boundary and then apply results and techniques of [1] to get uncountably many contractible open 4-manifolds. The main results are as follows:

THEOREM 1. *There exist countably many different contractible 2-complexes P_i with regular neighborhoods $N_i^4 \subset S^4$ such that for every i:*

(1) $N_i^4 \times I \approx I^5$,

(2) $\pi_1(BdN_i^4) \neq 1$,

(3) $\pi_1(S^4 - P_i) \neq 1$;

and if $i \neq j$:

(4) $\pi_1(BdN_i^4) \neq \pi_1(BdN_j^4)$ *and hence* $N_i^4 \neq N_j^4$ *and* $\text{int}\, N_i^4 \neq \text{int}\, N_j^4$,

(5) $\pi_1(S^4 - P_i) \neq \pi_1(S^4 - P_j)$ *and hence* $S^4 - P_i \neq S^4 - P_j$.

THEOREM 2. *For $n \geq 4$ there exist countably many different contractible $(n-2)$-complexes P_i^{n-2} with regular neighborhoods $M_i^n \subset S^n$ such that for every i:*

(1) $M_i^n \times I \approx I^{n+1}$,

(2) $\pi_1(BdM_i^n) \neq 1$,

(3) $\pi_1(S^n - P_i^{n-2}) \neq 1$;

and if $i \neq j$

(4) $\pi_1(S^n - P_i^{n-2}) \neq \pi_1(S^n - P_j^{n-2})$.

THEOREM 3. *There exist uncountably many contractible open 4-manifolds.*

THEOREM 4. *There exist uncountably many different involutions of E^4 any two distinguished by the fact that their fixed point sets are manifolds having non-isomorphic fundamental groups.*

§2. DEFINITIONS

We will use the standard terminology I^n, E^n, and S^n for the unit n-cell, Euclidean n-space and the n-sphere respectively. If M is an n-manifold, then int M and BdM will denote the interior and boundary of M, respectively. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. Topological equivalence will be denoted by $=$, and we will use $\approx$ to denote combinatorial equivalence. We will use techniques and terminology similar to that of Whitehead [8] or Zeeman [9]. Also we will use the notions of infinite sums of manifolds and group systems as in [1].

§3. PRELIMINARIES

We will want to make use of the main result Theorem (4.1) of [1], so for convenience we will quote it here.

THEOREM 0. *Let M and N be infinite connected sums of compact n-manifolds with non-empty connected boundaries. Then if* int $M =$ int N, $\pi_1(BdM) = \pi_1(BdN)$ *for* $n \geq 4$.

For each n, let K_n be the contractible 2-complex formed by attaching a disk D to a circle α by the formula $\alpha^{n+1}\alpha^{-n}$. By using the same techniques as those in [5] or [9], K_n can be embedded in a contractible 4-manifold W_n^4 with boundary so that W_n^4 can be considered as a regular neighborhood of K_n. Briefly let T^3 be a solid 3-dimensional torus forming half of the boundary of $S^1 \times I^3$ in E^4. For each n, we consider a certain embedding of a simple closed curve Γ_n in int T^3 so that:

(1) Γ_n is homotopic but not isotopic to the core $S^1 \times \{0\} \subset S^1 \times I^2 = T^3$;

(2) under the natural map $\rho : S^1 \times I^2 \to S^1 \times \{0\}$, $\rho(\Gamma_n)$ wraps around $S^1 \times \{0\}$ $n+1$ times in a counterclockwise direction and n times in a clockwise direction;

(3) in forming the 4-manifold W_n^4 with boundary by attaching a 2-handle to $Bd(S^1 \times I^3)$ along the curve Γ_n, we will get $\pi_1(BdW_n^4) \neq 1$.

Also for simplicity in computing $\pi_1(BdW_n^4)$ we will assume that in adding the 2-handle, there is no "twisting" of the tubular neighborhood of Γ_n thus formed as we go around Γ_n.

For $n = 1$ the embedding of Γ_1 is just that given in [5] or [9]. For $n = 2$ the embedding of Γ_2 is indicated in the bottom illustration of Fig. 1. The general embedding of Γ_n for $n \geq 3$ is indicated in Fig. 2 where γ_n can be considered as the simple closed curve $\{x\} \times BdI^2 \subset S^1 \times I^2 = T^3$.

LEMMA 1. *W_n^4 can be given a combinatorial triangulation so that K_n can be considered as a subcomplex of W_n^4 and $W_n^4 \searrow K_n$.*

Proof. The proof follows essentially word for word to the proof of Theorem (5) of [9].

LEMMA 2. *$W_n^4 \times I \approx I^5$ and hence W_n^4 can be embedded in S^4.*

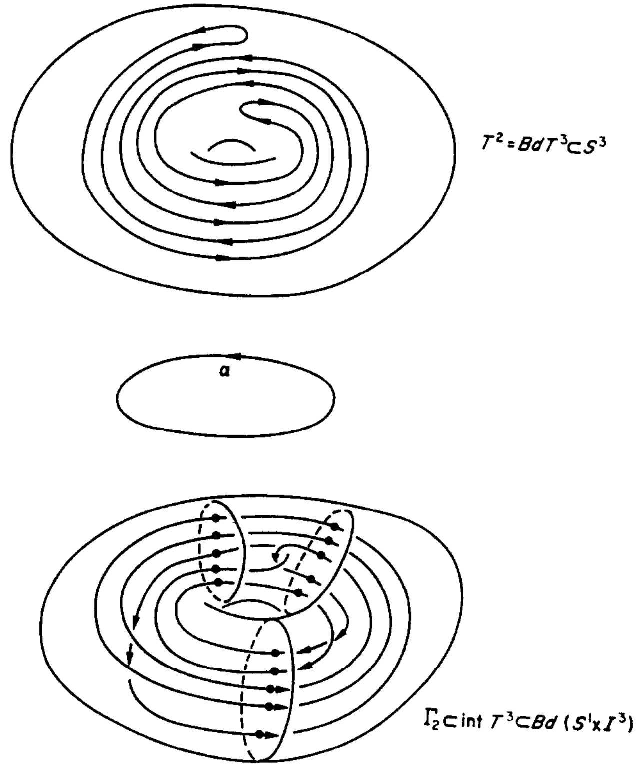

FIG. 1.

Proof. We observe that for each n K_n can be embedded in S^3 (for example if $n = 2$ consider the top illustration of Fig. 1 and think of S^3 as the union of two solid 3-dimensional tori) and hence can apply Lemma 2 of [4]. This is just a simple application of the corollary of [7]; that is, if K_n can be embedded in E^3 then the regular neighborhood of any embedding of K_n in E^5 is a combinatorial 5-cell.

LEMMA 3. *For each even n* $\pi_1(BdW_n^4) \neq 1$.

Proof. A presentation of $\pi_1(BdW_n^4)$ can be found by looking at the fundamental group of $E^3 - (\Gamma_n + \gamma_n)$ as indicated in Fig. 2 and adding the relations corresponding to curves slightly above each of Γ_n and γ_n respectively.

The resulting group has the following presentation:
generators: x and y
relations: $w = (xy)^{n+1}x(\bar{y}\bar{x})^{n+1}$

$I.$ $\bar{x}(w\bar{y})^n w(y\bar{w})^n = 1$

$II.$ $(xy)^{n+1}y(\bar{y}\bar{x})^{n+1}\bar{y}(w\bar{y})^n\bar{w}(y\bar{w})^n w = 1$

γ_n: $\bar{y}^{2n+1}(y\bar{w})^n y(xy)^n x = 1$

Γ_n: $(w\bar{y})^n(xy)^{n+1} = 1$

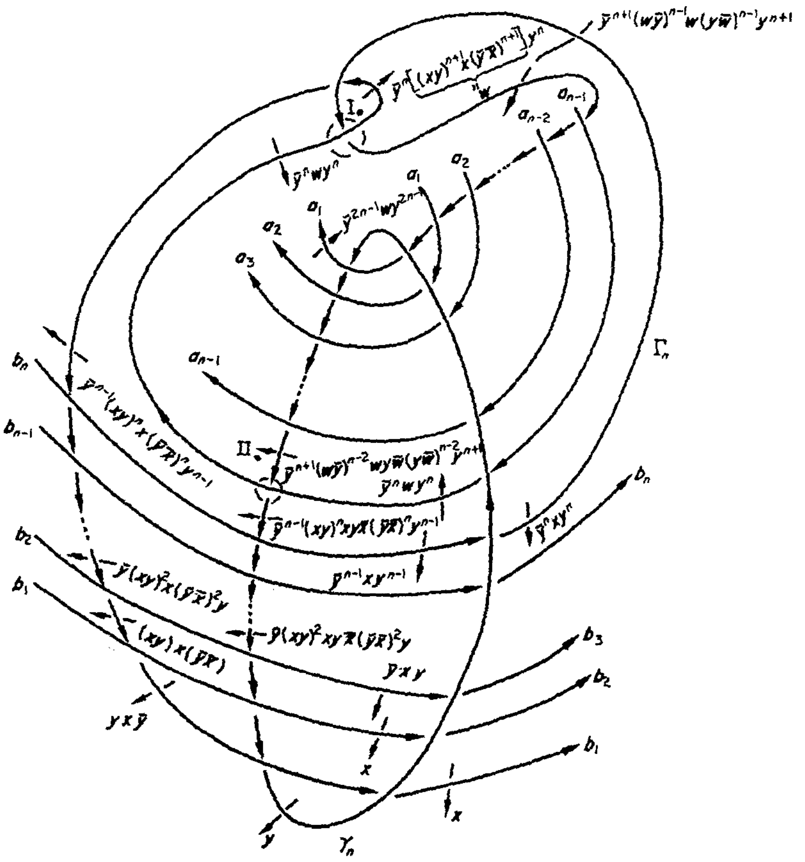

FIG. 2.

Using I. and $w = (xy)^{n+1}x(\bar{y}\bar{x})^{n+1}$ in II. we get that $1 = 1$. Using the fact that Γ_n: gives $(xy)^{n+1} = (y\bar{w})^n$, relation I. is equivalent to $w = (xy)^{n+1}x(\bar{y}\bar{x})^{n+1}$. Hence we get that $\pi_1(BdW_n^4)$ has the following presentation:

$$G_n = \{x,\ y | y^{2n+2} = (xy)^{n+1}y(xy)^{n+1},\ (xy)^{n+1} = [y(xy)^{n+1}\bar{x}(\bar{y}\bar{x})^{n+1}]^n\}$$

Adding the relation $(xy)^{n+1} = 1$, we get:

$$G_n' = \{x,\ y | (xy)^{n+1} = y^{2n+1} = (y\bar{x})^n = 1\}.$$

Setting $xy = \beta$ and $y^2 = \alpha$, we get the group:

$$G_n'' = \{\alpha,\ \beta | \beta^{n+1} = \alpha^{2n+1} = (\alpha\bar{\beta})^n = 1\}.$$

If we only consider the groups for n even, we can add the relation $(\alpha\bar{\beta})^2 = 1$ and hence we have:

$$\hat{G}_n = \{\alpha,\ \beta | \beta^{n+1} = \alpha^{2n+1} = (\alpha\bar{\beta})^2 = 1\}.$$

$\hat{G}_n$ can be shown to have a non-trivial representation in the alternating group A_{2n+1} by sending $\alpha \to (123 \ldots 2n\ 2n+1)$ and $\beta \to (135 \ldots 2n-1\ 2n+1)$.

For each even n we have a map ψ'_n taking G_n onto $\hat{G}_n$ and a map ψ''_n taking $\hat{G}_n$ into A_{2n+1}. Let $\hat{A}_{2n+1} = \psi''_n(\hat{G}_n)$ be the subgroup of A_{2n+1} generated by $(123\ldots 2n\; 2n+1)$ and $(135\ldots 2n-1\; 2n+1)$. Let $\psi_n = \psi''_n \cdot \psi'_n$ be the map taking G_n onto $\hat{A}_{2n+1}$. We note that $\hat{A}_{2n+1}$ is not Abelian.

LEMMA 4. *The sequence $G_2, G_4, G_6, \cdots$ contains infinitely many distinct groups.*

Proof. Let $\phi : Z^+ \to Z^+$ be the map of the positive integers defined by $\phi(n) = (2n+1)!$ The proof is now similar to Lemma (5.3) of [1]. That is, we will show that $G_n, G_{\phi(n)}, G_{\phi^2(n)}, \cdots$ are all distinct by showing that there is no surjective map

$$G_{\phi^j(n)} \to G_{\phi^i(n)}$$

whenever $i < j$.

In using the proof given in [1] we first look at G_n, $G_{\phi(n)}$ as was done there. We note that if $\eta : G_{(2n+1)!} \to G_n$ is surjective, then $\psi_n \circ \eta : G_{(2n+1)!} \to \hat{A}_{2n+1}$ is surjective. Also we use the fact that y and xy generate $G_{(2n+1)!}$ and hence $v = \psi_n \eta(y)$ and $u = \psi_n \eta(xy)$ generate $\hat{A}_{2n+1}$. But in considering the relations defining $G_{(2n+1)!}$ we get that

$$u^{(2n+1)!+1} = [v\, u^{(2n+1)!+1} v\, \bar{u}^{(2n+1)!+2}]^{(2n+1)!}.$$

Since the order of A_{2n+1} containing $\hat{A}_{2n+1}$ is $(2n+1)!/2$ we get that $u = 1$ and hence we get the contradiction that the non Abelian group $\hat{A}_{2n+1}$ is generated by one element. The remainder of the proof is similar to that given in [1], except using the notation given above.

§4. MAIN RESULTS

Proof of Theorem 1. Applying Lemmas 1–4 we have for the appropriate choices of the integers i (for example, integers of the form $\{\phi^j(n)\}\; j = 0, 1, 2, \cdots$) countably many different contractible 2-complexes $K_i \subset W_i^4 \subset E^4$ such that $W_i^4 \times I \approx I^5$ for every i and if $i \neq j$, $\pi_1(BdW_i^4) \neq \pi_1(BdW_j^4) \neq 1$. Let P_i be the contractible 2-complex formed as the union of two copies of K_i in $2W_i^4 \approx S^4$ plus a polygonal arc A_i intersecting each copy of K_i and BdW_i^4 in a single point, respectively. Then we have that $\pi_1(S^4 - P_i) = \pi_1(BdW_i^4) \neq 1$ since $S^4 - (K_i + K_i) \approx BdW_i^4 \times (-1, 1)$ and hence $\pi_1(BdW_i^4) = \pi_1(S^4 - (K_i + K_i)) = \pi_1(S^4 - P_i)$. Also the regular neighborhood N_i^4 of P_i in S^4 is the sum $W_i^4 \# W_i^4$ (as in [1]) and hence $\pi_1(BdN_i^4) \cong \pi_1(BdW_i^4) * \pi_1(BdW_i^4)$. The fact that $N_i^4 \times I \approx I^5$ follows since $W_i^4 \times I \approx I^5$ and $N_i^4 \times I$ is just two copies of $W_i^4 \times I$ identified along a 4-cell in the boundary of each.

Proof of Theorem 2. Let B^4 be a combinatorial 4-cell in S^4 such that $N_i^4 \subset \operatorname{int} B^4$. Then $\pi_1(S^4 - P_i) = \pi_1(B^4 - P_i) \neq 1$. Let $\mathcal{S}P_i$ be the contractible 3-complex gotten from the suspension of P_i (say from points p and q) embedded in S^5 where $S^5 \approx \mathcal{S}B^4 \cup \mathcal{C}(Bd\mathcal{S}B^4)$ with $\mathcal{C}(Bd\mathcal{S}B^4)$ the cone over the boundary of the 5-cell $\mathcal{S}B^4$ (say from the point r). Using this embedding of $\mathcal{S}P_i$ in S^5 we have that $S^5 - (\mathcal{S}P_i + \overset{\frown}{prq})$ is of the same homotopy type as $B^4 - P_i$. Hence $\pi_1(S^5 - \mathcal{S}P_i) = \pi_1(S^5 - (\mathcal{S}P_i + \overset{\frown}{prq})) = \pi_1(S^4 - P_i)$.

Let M_i^5 be the regular neighborhood of $\mathscr{S}P_i$ in S^5. Since $\pi_1(S^5 - \mathscr{S}P_i) \neq 1$ it follows from [3] that $\pi_1(BdM_i^5) \neq 1$. It was shown in [2] that under the above conditions $M_i^5 \times I \approx I^6$. Hence by taking repeated suspensions we get the conclusions of Theorem 2.

Proof of Theorem 3. This follows exactly in the same way similar results were obtained in [1] for $n \geq 5$. That is, we consider the sequence of groups $G_n, G_{\phi(n)}, G_{\phi^2(n)}, \ldots$ representing presentations of $\pi_1(BdW_n^4)$, $\pi_1(BdW_{\phi(n)}^4)$, $\pi_1(BdW_{\phi^2(n)}^4)$, ..., respectively. They are all distinct (by Lemma 4) and indecomposable (each $G_{\phi^i(n)}$ is a perfect group on two generators). By forming infinite sums W_α of the W_i^4's (as in [1]) in uncountably many different ways such that in two different ones some W_i^4 occurs more in one than in the other, Theorem 0 applies to give Theorem 3 (for a discussion of group systems and how we get $\pi_1(W_\alpha)$ refer to [1]).

Proof of Theorem 4. Since each $W_i^4 \times I \approx I^5$, it follows easily that an infinite sum W_α of any sequence of W_i^4's has the property that $2W = R^4$. Again an application of the notion of the group systems of an infinite sum (as in [1]) gives Theorem 4.

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